Mortgage Repayment Formula Derivation

OK, so you've found your perfect home, but there's one snag; you're £100,000 short. You therefore need to take out a *mortgage* from a bank, which is a *securitised loan* from a bank which you pay off over a period of years (usually 20 or 25 years). The bank is willing to loan you such a large amount of money because if you fail to pay then the bank takes the property from you to recover its losses. You also pay *interest* on the mortgage for the duration of the loan, which makes the transaction attractive to the bank.

Once the mortgage has been taken, you pay the bank back in monthly instalments. These payments have to pay off the interest which has accumulated on the debt during the course of the month. If you *only* paid off the interest, then you would never pay off the debt itself, so a *repayment mortgage* requires you to pay an additional amount on top of the interest which reduces the outstanding debt. Because the debt has been reduced there will therefore be a lower amount interest accrued in the following month. And so on…

The question is "How do I calculate the amount the I will be paying back per month, given the interest rate, mortgage length, and size of loan if I want to have constant monthly payments?"

Annual Repayment Formula

Let us suppose you take a £100,000 mortgage repayable over 25 years at 5% interest. For ease of derivation we will start with annual repayments, even though monthly repayments are more normal. For the first two years we have:

Year	Debt at start of year (£)	Interest accumulated by end of year (£)	Capital repayment (£)
1	$100,000$	$\frac{5}{100} \times 100,000$	X
2	$100,000 - X$	$\frac{5}{100} \times (100,000 - X)$	Y
$\vdots$	$\vdots$	$\vdots$	$\vdots$

In general, for a loan of £D at R% interest we have:

Year	Debt at start of year (£)	Interest accumulated by end of year (£)	Capital repayment (£)
1	D	$\frac{DR}{100}$	X
2	$D - X$	$\frac{(D-X)R}{100}$	Y
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Now, if we want the overall annual payment of interest and capital to be constant we must have:

$$\begin{aligned}\frac{DR}{100} + X &= \frac{(D-X)R}{100} + Y \\ \frac{DR}{100} + X &= \frac{DR}{100} - \frac{RX}{100} + Y \\ X &= Y - \frac{RX}{100} \\ Y &= \left(1 + \frac{R}{100}\right)X.\end{aligned}$$

So we see that the second year's payment is $\left(1 + \frac{R}{100}\right)$ times by the first year's payment. Similarly the third year's payment is $\left(1 + \frac{R}{100}\right)$ times by the second year's payment. And so on…

Year	Debt at start of year (£)	Interest accumulated by end of year (£)	Capital repayment (£)
1	D	$\frac{DR}{100}$	X
2	$D-X$	$\frac{(D-X)R}{100}$	$\left(1+\frac{R}{100}\right)X$
3	$D-X-\left(1+\frac{R}{100}\right)X$	$\frac{(D-X-(1+\frac{R}{100})X)R}{100}$	$\left(1+\frac{R}{100}\right)^2 X$
4	$\vdots$	$\vdots$	$\left(1+\frac{R}{100}\right)^3 X$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
25	$\vdots$	$\vdots$	$\left(1+\frac{R}{100}\right)^{24} X$

i.e. the capital repayment column forms a geometric sequence with first term X and common ratio $(1+\frac{R}{100})$. The sum of the 25 capital repayments must equal the total mortgage amount D, so using the sum of a geometric series formula $S_n = a\left(\frac{r^n-1}{r-1}\right)$ we have:

$$\begin{aligned}
X + \left(1+\frac{R}{100}\right)X + \left(1+\frac{R}{100}\right)^2 X + \cdots + \left(1+\frac{R}{100}\right)^{24} X &= D\\
X\left(\frac{(1+\frac{R}{100})^{25}-1}{(1+\frac{R}{100})-1}\right) &= D\\
X\left(\frac{(1+\frac{R}{100})^{25}-1}{\frac{R}{100}}\right) &= D\\
X &= \frac{\frac{DR}{100}}{(1+\frac{R}{100})^{25}-1}\\
X &= \frac{DR}{100[(1+\frac{R}{100})^{25}-1]}
\end{aligned}$$

This obviously becomes

$$X = \frac{DR}{100\left[\left(1+\frac{R}{100}\right)^Y - 1\right]}$$

if the mortgage length is Y years.

So all we need to do now to discover the annual cost of the mortgage is to sum X (the capital repayment amount in the first year) and the interest accrued in the first year.

$$\begin{aligned}
\text{Annual Repayment} &= \frac{DR}{100} + \frac{DR}{100[(1+\frac{R}{100})^Y - 1]}\\
&= \frac{DR}{100}\left(1 + \frac{1}{(1+\frac{R}{100})^Y - 1}\right)\\
&= \frac{DR}{100}\left(\frac{(1+\frac{R}{100})^Y - 1}{(1+\frac{R}{100})^Y - 1} + \frac{1}{(1+\frac{R}{100})^Y - 1}\right)\\
&= \frac{DR}{100}\left(\frac{\left(1+\frac{R}{100}\right)^Y}{\left(1+\frac{R}{100}\right)^Y - 1}\right)
\end{aligned}$$

So

$$\boxed{\text{Annual Repayment} = \frac{DR}{100}\left(\frac{\left(1+\frac{R}{100}\right)^Y}{\left(1+\frac{R}{100}\right)^Y - 1}\right)}$$

So in our example of a £100,000 loan repayable over 25 years at 5% interest we have

$$\text{Annual Repayment} = \frac{100,000 \times 5}{100}\left(\frac{\left(1+\frac{5}{100}\right)^{25}}{\left(1+\frac{5}{100}\right)^{25}-1}\right) = 5000 \times \frac{1.05^{25}}{1.05^{25}-1} = £7,095.25$$

It is worth noting that on borrowing £100,000 you end up repaying $25 \times 7,095.25 = £177,381.25$ over the course of the mortgage(!)

Monthly Repayment Formula

The derivation for monthly repayments is very similar, except instead of $\frac{DR}{100}$ interest per *year* we have $\frac{D\times\frac{R}{12}}{100} = \frac{DR}{1200}$ per *month*. And instead of 25 or Y payments, we have 25×12 or $12Y$ payments.

So

$$\boxed{\text{Monthly Repayment} = \frac{DR}{1200}\left(\frac{\left(1+\frac{R}{1200}\right)^{12Y}}{\left(1+\frac{R}{1200}\right)^{12Y}-1}\right)}$$

So in our example of a £100,000 loan repayable over 25 years at 5% interest we have

$$\text{Monthly Repayment} = \frac{100,000 \times 5}{1200}\left(\frac{\left(1+\frac{5}{1200}\right)^{300}}{\left(1+\frac{5}{1200}\right)^{300}-1}\right) = £584.59$$

So here on borrowing £100,000 you end up repaying $300 \times 584.59 = £175,377$ over the course of the mortgage, which is slightly better than annual repayments.